ANALYTIC INVESTIGATION OF VIEWPORTS IN DEEP SUBMERGENCE STRUCTURES

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ABSTRACT

The basic parameters and boundary conditions for solution of the equilibrium equations for a viewport in a deep submersible are defined. A likely solution method is chosen and investigated in order to determine its applicability and to learn more of the behavior and inter-relationships of the governing equations. The difficulties encountered in the chosen solution method are explained and another approach is suggested.

THESIS SUPERVISOR: J.H. Evans

TITLE: Professor of Naval Architecture

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INTRODUCTION

Since Piccard's Trieste,[1] almost all deep submersibles have utilized small plexiglass viewports for observation and control. The governing structural mechanics equations for the viewports have not been solved and, therefore, the stress state and predicted deflections have not been known. The interaction with the pressure hull and its reinforcement has also mainly been known from finite element analyses.

The principal investigations of the behavior of plexiglass viewports were experimental tests reported by Stachiw.[2,3,4,5] These tests included short term loading of conical, flat, and spherical windows as well as long term creep of conical windows. The data from these tests provides sufficient information for safe, large safety factor design. It does not, however, provide insight into the stresses in the window support or detailed information of expected viewport performance and useful life.

Since plexiglass is inexpensive, the viewports have been greatly over-designed and the present state of the art has been adequate. However, for future, more sophisticated vehicles with larger viewports, a better

understanding of the stresses and deflections in the viewports and the resultant interaction with the pressure hull will permit a safer and less expensive design and greater insight into critical design aspects.

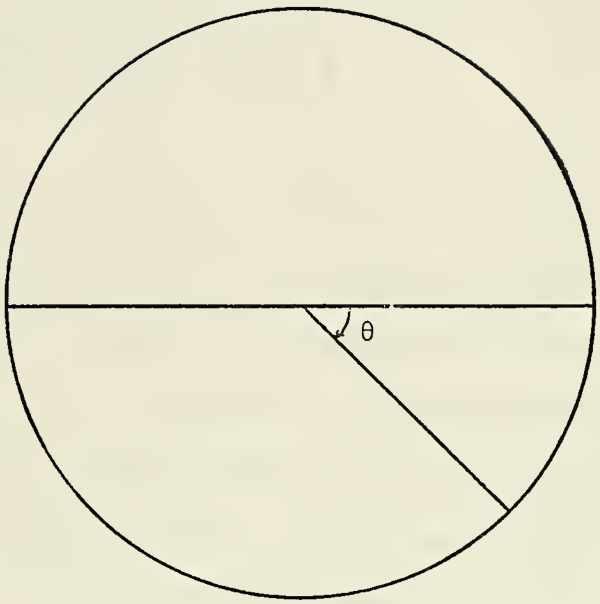
BACKGROUND

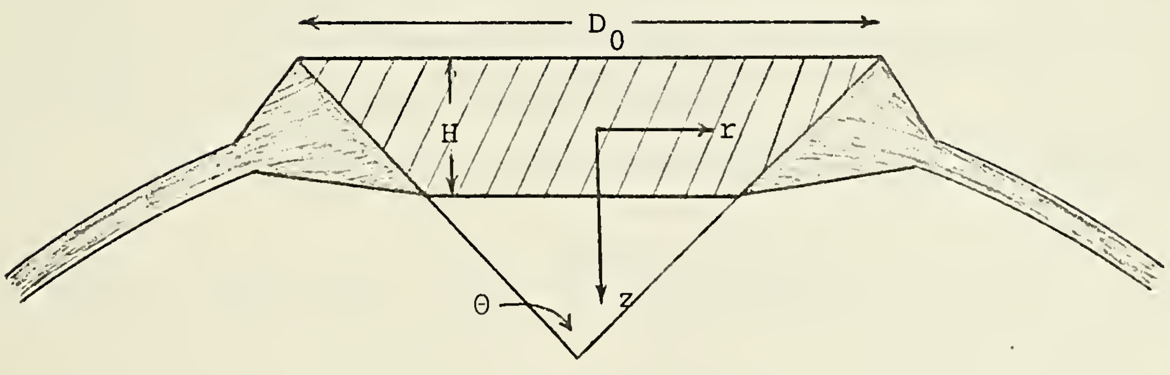
Most viewports presently utilized in undersea vehicles are truncated cones mounted in heavily reinforced openings in the pressure hull. They are held in position by mounting rings but the primary seal is simply from surface to surface contact. The contact surfaces are usually polished to ensure a good seal and a 90 degree cone angle is utilized. This angle was determined by Piccard to be the best design and has been utilized successfully by designers since then. The tests by Stachiw have supported the opinion that the 90° cone angle is best for acrylic viewports.

It can be seen that the viewport is similar to a thick, flat, circular plate with slanted ends. The stress distribution is axi-symmetric and therefore circular coordinates should be utilized in forming the governing equations. For the coordinate system in Figure 1, the deformation is symmetrical around the z axis and therefore all derivatives with respect to θ are zero and shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ are zero.

Timoshenko and Goodier[6] show that the equations of equilibrium therefore take the form:

Figure 1





$$\frac{\delta\sigma_r}{\delta r} + \frac{\delta\tau_{rz}}{\delta z} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad (1)$$

$$\frac{\delta\tau_{rz}}{\delta r} + \frac{\delta\sigma_z}{\delta z} + \frac{\tau_{rz}}{r} = 0$$

At present most submersibles have moderate depth capabilities and therefore do not need excessively thick viewports. The desire for good visibility therefore results in a design with a thickness to diameter ratio of usually less than .5 for the viewport. The fact that Stachiw[2] only tested viewports with thickness to inner diameter ratios of .631 or less illustrates that this is presently the major area of concern. Consequently, the compatibility equations can therefore be simplifed by assuming plane stress. Although this is not completely accurate, it should allow an accurate prediction of the behavior of almost all desirable viewport designs.

The compatibility relations therefore take the form:[6]

$$\nabla^2\sigma_r - \frac{2}{r^2}(\sigma_r-\sigma_\theta) + \frac{1}{1+\nu}\,\frac{\delta^2(\sigma_r+\sigma_\theta+\sigma_z)}{\delta r^2} = 0$$

$$\nabla^2\sigma_\theta + \frac{2}{r^2}(\sigma_r-\sigma_\theta) + \frac{1}{1+\nu}\,\frac{1}{r}\,\frac{\delta(\sigma_r+\sigma_\theta+\sigma_z)}{\delta r} = 0 \qquad (2)$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\,\frac{\delta^2(\sigma_r+\sigma_\theta+\sigma_z)}{\delta z^2} = 0$$

$$\nabla^2\tau_{rz} - \frac{1}{r^2}\tau_{rz} + \frac{1}{1+\nu} \cdot \frac{\delta^2(\sigma_r+\sigma_\theta+\sigma_z)}{\delta r\delta z} = 0$$

The strain components, therefore, become:

$$\varepsilon_r = \frac{\delta u}{\delta r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\delta w}{\delta z}$$

$$\gamma_{rz} = \frac{\delta u}{\delta z} + \frac{\delta w}{\delta r}$$

Equations (1) and (2) can be satisfied by expressing the stresses in terms of a stress function ϕ.[6] The stresses, therefore, become:

$$\sigma_r = \frac{\delta}{\delta z}(\nu\nabla^2\phi - \frac{\delta^2\phi}{\delta r^2})$$

$$\sigma_\theta = \frac{\delta}{\delta z}(\nu\nabla^2\phi - \frac{1}{r}\frac{\delta\phi}{\delta r})$$

$$\sigma_z = \frac{\delta}{\delta z}[(2-\nu)\nabla^2\phi - \frac{\delta^2\phi}{\delta z^2}]$$

$$\tau_{rz} = \frac{\delta}{\delta r}[(1-\nu)\nabla^2\phi - \frac{\delta^2\phi}{\delta z^2}]$$

subject to the condition that:

$$\nabla^2\nabla^2\phi = 0 \qquad (3)$$

These equations apply to a viewport in an undersea structure or pressure facility provided the appropriate boundary conditions are met. The first and most obvious boundary condition is that the normal stress on the pressurized face is equal to and opposing the external pressure. Obviously, the condition is reversed on the unpressurized face where the normal stress is zero. The shear stress on the flat faces is also zero since they are free.

The boundary conditions on the slanted ends are less obvious. In reality there will be a contraction and rotation of the mounting reinforcement with pressure. However, since the reinforcement is very thick steel with a high modulus of elasticity, it can be safely considered as being rigid compared to the plexiglass. Since the viewport is not physically attached to its mounting, it is free to slide along the mounting opposed only by friction and the shape of the mounting.

Two boundaries on the behavior of the viewport at its mounting can be specified. The viewport can be considered to be rigidly fixed to its mounting or it can be allowed to slide freely. In the first case, it seems that the stresses produced will be higher than the actual situation and the displacements will be lower. In the second case, it seems that the stresses produced will be

slightly lower than the actual situation and the displacements will be higher.

The first case is satisfied if all displacements are zero at the boundary, i.e.

$$u = w = 0 \text{ at } r = \frac{D_0}{2} - (\frac{H}{2} + z) \tan \theta$$

The second case is satisfied if the displacement normal to the mounting is zero at the boundary, i.e.

$$u \cos \theta + w \sin \theta = 0 \text{ at } r = \frac{D_0}{2} - (\frac{H}{2} + z) \tan \theta$$

In seeking solutions of equations (1), (2), and (3), Timoshenko and Goodier considered a polynomial form of the stress function ϕ which also satisfied the equation:

$$\frac{\delta^2\phi}{\delta R^2} + \frac{2}{R}\frac{\delta\phi}{\delta R} + \frac{1}{R^2} \operatorname{ctn} \Psi \frac{\delta\phi}{\delta\Psi} + \frac{1}{R^2}\frac{\delta^2\phi}{\delta\Psi^2} = 0$$

This resulted in particular solutions of the form:

$$\phi_n = a_0[z^n - \frac{n(n-1)}{2(2n-1)} R^2 z^{(n-2)} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} R^4 z^{(n-4)} - \ldots]$$

and additional solutions of equations (1) and (2) found by multiplying the particular solution by R^2 where $R^2 = r^2 + z^2$. Consequently, the following possible

solutions can be found:

$$\phi_0 = a_0$$

$$\phi_1 = a_1 z$$

$$\phi_2 = a_2[2z^2 - r^2]$$

$$\phi_3 = a_3[2z^3 - 3r^2 z]$$

$$\phi_4 = a_4[8z^4 - 24r^2 z^2 + 3r^4]$$

$$\phi_5 = a_5[8z^5 - 40r^2 z^3 + 15r^4 z]$$

$$\phi_6 = a_6[\tfrac{16}{3}z^6 - 40z^4 r^2 + 30z^2 r^4 - \tfrac{5}{3}r^6]$$

and

$$\phi_2 = b_2[r^2 + z^2]$$

$$\phi_3 = b_3[z^3 + zr^2]$$

$$\phi_4 = b_4[2z^4 + r^2 z^2 - r^4]$$

$$\phi_5 = b_5[2z^5 - r^2 z^3 - 3r^4 z]$$

$$\phi_6 = b_6[8z^6 - 16z^4 r^2 - 21z^2 r^4 + 3r^6]$$

Timoshenko and Goodier considered a flat circular plate loaded with a uniform pressure on one face. The boundary conditions were:

$$\tau_{rz} = 0 \text{ at } z = \pm\frac{H}{2}$$

$$\sigma_z = 0 \text{ at } z = \frac{H}{2}$$

$$\sigma_z = -P \text{ at } z = -\frac{H}{2}$$

These are identical to all but the last boundary condition for a viewport.

By assuming a solution form of:

$$\phi = a_6(\frac{16}{3}z^6 - 40z^4r^2 + 30z^2r^4 - \frac{5}{3}r^6)$$

$$+ b_6(8z^6 - 16z^4r^2 - 21z^2r^4 + 3r^6)$$

$$+ a_4(8z^4 - 24r^2z^2 + 3r^4)$$

$$+ a_3(2z^3 - 3r^2z)$$

$$+ b_3(z^3 + zr^2)$$

they found that

$$a_6 = (\frac{18 - 11\nu}{3520H^3}) P$$

$$b_6 = (\frac{1}{352H^3}) P$$

$$a_4 = (-\frac{3}{384H})\ P$$

$$a_3 = (\frac{-1+5\nu}{72+60\nu})\ P$$

$$b_3 = (-\frac{1}{24+20\nu})\ P$$

Therefore the stresses become:

$$\sigma_r = P[(2+\nu)\frac{z^3}{H^3} - \frac{3(3+\nu)}{4}\frac{r^2 z}{H^3} - \frac{3}{4}\frac{z}{H}]$$

$$\sigma_z = P[-\frac{z^3}{2H^3} + \frac{3}{2}\frac{z}{H} - \frac{1}{2}]$$

$$\tau_{rz} = 3Pr\ (\frac{H^2}{4} - z^2)$$

Using the equations for strains given previously we find that:

$$u = -\frac{1+\nu}{E}\ \frac{\delta^2 \phi}{\delta r \delta z}$$

$$w = \frac{1+\nu}{E}\ [2(1-\nu)\frac{\delta^2\phi}{\delta r^2} + 2(1-\nu)\ \frac{1}{r}\ \frac{\delta\phi}{\delta r} + (1-2\nu)\ \frac{\delta^2\phi}{\delta z^2}]$$

Consequently,

$$u = \frac{1+\nu}{E}[(\frac{2-\nu}{H^3})\ z^3 r + (\frac{-3+3\nu}{4H^3})\ zr^3 + (\frac{-3}{4H})\ zr + (\frac{5\nu}{12+10\nu})\ r]\ P$$

$$w = \frac{1+\nu}{E}[(\frac{-1-\nu}{2H^3})z^4+(\frac{3\nu}{2H^3})z^2r^2+(\frac{3-3\nu}{16H^3})r^4+(\frac{3}{4H})z^2+(-\frac{3}{8H})r^2$$

$$+ (\frac{-5}{12+10\nu})z] \; P$$

The final boundary condition for the first case for the viewport requires that u = w = 0 at $r = \frac{D_0}{2} - (\frac{H}{2} + z)\tan\theta$. The flat plate solution, however, gives

$$u = \frac{1+\nu}{E} P \; [(\frac{-2+\nu}{H^3})z^4 \tan\theta + (\frac{3-3\nu}{4H^3})z^4 \tan^3\theta$$

$$+ (\frac{2-\nu}{2H^3}) \Delta z^3 + (\frac{-9+9\nu}{8H^3}) \Delta z^3 \tan^2\theta$$

$$+ (\frac{9-9\nu}{16H^3}) \Delta^2 z^2 \tan\theta + (\frac{3}{4H}) z^2 \tan\theta$$

$$+ (\frac{-3+3\nu}{32H^3}) \Delta^3 z + (\frac{-3}{8H}) \Delta z + (\frac{-5\nu}{12+10\nu}) z \tan\theta$$

$$+ (\frac{5\nu}{24+20\nu}) \Delta] = 0$$

$$w = \frac{1+\nu}{E} P \; [(\frac{-1-\nu}{2H^3}) z^4 + (\frac{3\nu}{2H^3}) z^4 \tan^2\theta$$

$$+ (\frac{3-3\nu}{16H^3}) z^4 \tan^4\theta + (\frac{-3\nu}{2H^3}) \Delta z^3 \tan\theta$$

$$+ \left(\frac{-3+3\nu}{8H^3}\right) \Delta z^3 \tan^3\theta + \left(\frac{3}{4H}\right) z^2 + \left(\frac{3\nu}{8H^3}\right) \Delta^2 z^2$$

$$+ \left(\frac{-3}{8H}\right) z^2 \tan^2\theta + \left(\frac{9-9\nu}{32H^3}\right) \Delta^2 z^2 \tan^2\theta$$

$$+ \left(\frac{-5}{12+10\nu}\right) z + \left(\frac{3}{8H}\right) \Delta z \tan\theta + \left(\frac{-3+3\nu}{32H^3}\right) \Delta^3 z \tan\theta$$

$$+ \left(\frac{-3}{32H}\right) \Delta^2 + \left(\frac{3-3\nu}{256H^3}\right) \Delta^4$$

The final boundary condition for the second case for the viewport requires that $u + w \tan\theta = 0$ at $r = [\frac{D_0}{2} - (\frac{H}{2} + z) \tan\theta]$. The flat plate solution, however, gives

$$u + w \tan\theta = \frac{1+\nu}{E} P[\left(\frac{-5+\nu}{2H^2}\right) \tan\theta\, z^4 + \left(\frac{3+3\nu}{4H^3}\right) \tan^3\theta\, z^4$$

$$+ \left(\frac{3-3\nu}{16H^3}\right) \tan^5\theta\, z^4 + \left(\frac{2-\nu}{2H^3}\right) \Delta z^3 + \left(\frac{-9-3\nu}{8H^3}\right) \Delta \tan^2\theta\, z^3$$

$$+ \left(\frac{-3+3\nu}{8H^3}\right) \Delta \tan^4\theta\, z^3 + \left(\frac{9-3\nu}{16H^3}\right) \Delta^2 \tan\theta\, z^2 + \left(-\frac{3}{8H}\right) \tan^3\theta\, z^2$$

$$+ \left(\frac{3}{2H}\right) \tan\theta z^2 + \left(\frac{9-9\nu}{32H^3}\right)\Delta^2\tan^3\theta z^2 + \left(\frac{-3+3\nu}{32H^3}\right)\Delta^3 z + \left(\frac{-3+3\nu}{32H^3}\right)\Delta^3\tan^2\theta z$$

$$+ \left(-\frac{3}{8H}\right)\Delta z + \left(\frac{3}{8H}\right)\Delta\tan^2\theta z + \left(\frac{-5}{12+10}\right) \tan\theta z$$

$$+ \left(\frac{3-3\nu}{256H^3}\right)\Delta^4\tan\theta + \left(-\frac{3}{32H}\right)\Delta^2\tan\theta + \left(\frac{5\nu}{24+20\nu}\right)\Delta \,]$$

where $\Delta = (D_0 - H\tan\theta)$.

Consequently, it can be seen that if these boundary conditions could also be satisfied, solutions would result which would be dependent on the material properties ν and E as well as the dimensions D_0, H, and θ and the pressure P. Since plexiglass is a viscoelastic material the properties ν and E will be slightly non-linear and will be functions of time. In practice, all viewports are designed to remain well within the elastic range since vision would be greatly distorted even before reaching the non-linear response. Consequently, a solution assuming that the material is linearly elastic will be adequate since the variation in properties of different batches of plexiglass makes more exact calculations of little use.

SOLUTION WITH SIXTH ORDER POLYNOMIAL ASSUMPTION

In order to satisfy either of these additional boundary conditions it is apparent that supplemental sixth order terms will be required. Since no other terms are possible using the method of Timoshenko and Goodier, other forms must be found. The desired forms must satisfy equation (3) and two possibilities therefore become apparent. Either $\nabla^2\nabla^2\phi = 0$ or $\nabla^2\phi = 0$ will satisfy equations (1) and (2).

Probable desired terms include sixth, fourth, and third order terms. General terms can be constructed as follows:

$$\phi = az^6 + bz^4r^2 + cz^2r^4 + dr^6$$

$$\phi = ez^4 + fz^2r^2 + gr^4$$

$$\phi = pz^3 + qzr^2$$

In order to satisfy $\nabla^2\nabla^2\phi = 0$ we find:

$$a = -\frac{12b + 8c}{45}$$

$$d = -\frac{3b + 8c}{72}$$

$$e = -\frac{8}{3}g - \frac{2}{3}f$$

Additional terms can be constructed as follows:

$$\phi = hz^6 + iz^4r^2 + jz^2r^4 + kr^6$$

$$\phi = lz^4 + mz^2r^2 + nr^4$$

$$\phi = sz^2 + tr^2$$

In order to satisfy $\nabla^2\phi = 0$ we find:

$$h = -\frac{16}{5}k$$

$$i = 24k$$

$$j = -18k$$

$$m = -8n$$

$$l = \frac{8}{3}n$$

The boundary conditions for the combination of the flat plate solution and the additional terms therefore become:

$$\tau_{rz} = 0 \text{ at } z = \pm\frac{H}{2}$$

$$\sigma_z = 0 \text{ at } z = \frac{H}{2}$$

$$\sigma_z = -P \text{ at } z = -\frac{H}{2}$$

$$u = w = 0 \text{ or } u + w\tan\theta = 0 \text{ at } r = \frac{1}{2}[D_0 - (H+2z)\tan\theta]$$

The boundary conditions for the additional terms alone therefore become:

$$\tau_{rz} = 0 \text{ at } z = \pm\frac{H}{2}$$

$$\sigma_z = 0 \text{ at } z = \pm\frac{H}{2}$$

$$u + u_{\text{flat plate}} = w + w_{\text{flat plate}} = 0 \text{ at } r = \frac{D_0}{z} - (\frac{H}{2} + z)\tan\theta$$

or

$$u + w\tan\theta + (u + w\tan\theta)_{\text{flat plate}} = 0 \text{ at } r = \frac{D_0}{z} - (\frac{H}{2} + z)\tan\theta$$

For the additional terms to satisfy the first boundary condition we find:

$$\tau_{rz} = 0 = (8c - 8\nu c - 6\nu b - 144k)\, H^2 r$$

$$+ (-6b + 6\nu b - 16c + 8\nu c + 144k)\, r^3$$

$$+ (32g - 32\nu g - 4\nu f + 32n)\, r$$

Since the constants b, c, k, g, f, and n are independent of r, the r^3 and r terms must each equal zero. Therefore

$$144k = 6b - 6\nu b + 16c - 8\nu c$$

and

$$32n = (6b + 8c)H^2 + (-32g + 32\nu g + 4\nu f)$$

Considering the second boundary condition we require that:

$$\sigma_z = 0 = (-4b - \frac{16}{3} c)(\pm H^3) + (8q - 4\nu q + 6p - 6\nu p)$$

Therefore

$$6p\ (1-\nu) = -8q + 4\nu q$$

and

$$b = -\frac{4}{3}c$$

At the boundary $r = \frac{D_0}{2} - (\frac{H}{2} + z) \tan\theta$

$$u = \frac{1+\nu}{E} [(-16b + 8\nu b - \frac{64}{3}c + \frac{32}{3}\nu c)\ z^4 \tan\theta$$

$$+ (6b - 6\nu b + 8c - 8\nu c)\ z^4 \tan^3\theta$$

$$+ (8b - 4\nu b + \frac{32}{3}c - \frac{16}{3}\nu c)\ \Delta z^3$$

$$+ (-9b + 9\nu b -12c + 12\nu c)\Delta^2 z^3 \tan^2\theta + (6b + 8c)H^2 z^2 \tan\theta$$

$$+ (\frac{9}{2}b - \frac{9}{2}\nu b + 6c - 6\nu c)\Delta^2 z^2 \tan\theta$$

$$+ (-32g + 32\nu g - 4f + 4\nu f)z^2 \tan\theta$$

$$+ (-\frac{3}{4}b + \frac{3}{4}\nu b - c + \nu c)\Delta^3 z + (-3b - 4c)H^2 \Delta z$$

$$+ (16g - 16\nu g + 2f - 2\nu f)\Delta z + (-2q)\ z \tan\theta + (q)\ \Delta\ ]$$

and

$$w = \frac{1+\nu}{E}\ [(4b + 4\nu b + \frac{16}{3}c + \frac{16}{3}\nu c)\ z^4$$

$$+(-12\nu b - 16\nu c)z^4 \tan^2\theta + (-\frac{3}{2}b + \frac{3}{2}\nu b - 2c + 2\nu c)z^4 \tan^4\theta$$

$$+(12\nu b + 16\nu c)\Delta z^3 \tan\theta + (3b - 3\nu b + 4c - 4\nu c)\Delta z^3 \tan^3\theta$$

$$+(-3\nu b - 4\nu c)\Delta^2 z^2 + (-\frac{9}{4}b + \frac{9}{4}\nu b - 3c + 3\nu c)\Delta^2 z^2 \tan^2\theta$$

$$+(-6b - 8c)H^2 z^2 + (4\nu f + 32\nu g)z^2 + (3b + 4c)H^2 z^2 \tan^2\theta$$

$$+(16g - 16\nu g + 2f - 2\nu f)z^2 \tan^2\theta + (-3b - 4c)H^2 \Delta z \tan\theta$$

$$+ \left(\frac{3}{4} b - \frac{3}{4} \nu b + c - \nu c\right) \Delta^3 z \tan\theta$$

$$+ (-16g + 16\nu g - 2f + 2\nu f) \Delta z \tan\theta + (6p - 12\nu p + 8q - 8\nu q) z$$

$$+ \left(-\frac{3}{32} b + \frac{3}{32} \nu b - \frac{1}{8} c + \frac{1}{8} \nu c\right) \Delta^4 + \left(\frac{3}{4} b + c\right) H^2 \Delta^2$$

$$+ \left(4g - 4\nu c + \frac{1}{2} f - \frac{1}{2} \nu f\right) \Delta^2]$$

Substituting for p and b we find:

$$u = \frac{1+\nu}{E} [(-32g + 32\nu g - 4f + 4\nu f) z^2 \tan\theta$$

$$+ (16g - 16\nu q + 2f - 2\nu f) \Delta z \qquad (4)$$

$$+ (-2q) z \tan\theta + q \Delta]$$

and

$$w = \frac{1+\nu}{E} [(4\nu f + 32\nu g) z^2 + (16g - 16\nu g + 2f - 2\nu f) z^2 \tan^2\theta$$

$$+ (-16g + 16\nu g - 2f + 2\nu f) \Delta z \tan\theta + \frac{4\nu q}{1-\nu} z \qquad (5)$$

$$+ \left(4g - 4\nu g + \frac{1}{2} f - \frac{1}{2} \nu f\right) \Delta^2]$$

Consequently, it is apparent that neither of the final boundary conditions can possibly be satisfied for the third or fourth order terms using this solution form. This occurs because the first boundary condition causes the fourth order constants, f and g, to cancel in the second boundary condition. This produces a relationship between b and c which prevents them from contributing to the displacements u and w. However, the addition of higher order terms may allow all boundary conditions to be met.

SOLUTION WITH INFINITE ORDER POLYNOMIAL ASSUMPTION

In order to investigate this possibility we consider a stress function of order n. Therefore

$$\phi_n = a_n z^n + b_n z^{n-2} r^2 + c_n z^{n-4} r^4 + \ldots.$$

The terms contain only even powers of r since only even powers will satisfy equation (3). Therefore $\nabla^2\phi$ and $\nabla^2\nabla^2\phi$ may be expressed as:

$$\nabla^2\phi = [2\cdot 2b_n + n(n-1)a_n]z^{n-2} + [4\cdot 4c_n + (n-2)(n-3)b_n]z^{n-4}r^2$$
$$+ [6\cdot 6d_n + (n-4)(n-5)c_n]z^{n-6}r^4 + \ldots.$$

and

$$\nabla^2\nabla^2\phi = [2\cdot 2\cdot 4\cdot 4c_n + 2\cdot 2\cdot 2(n-2)(n-3)b_n$$
$$+ n(n-1)(n-2)(n-3)a_n]z^{n-4} + [4\cdot 4\cdot 6\cdot 6d_n + 2\cdot 4\cdot 4(n-4)(n-5)c_n$$
$$+ (n-2)(n-3)(n-4)(n-5)b_n]z^{n-6}r^2 + [6\cdot 6\cdot 8\cdot 8e_n + 2\cdot 6\cdot 6(n-6)(n-7)d_n$$
$$+ (n-4)(n-5)(n-6)(n-7)c_n]z^{n-8}r^4 + \ldots.$$

Consequently, in order to satisfy $\nabla^2\nabla^2\phi = 0$ we find:

$$c_n = -\frac{2(n-2)(n-3)}{4\cdot 4} b_n - \frac{n(n-1)(n-2)(n-3)}{2\cdot 2\cdot 4\cdot 4} a_n$$

$$d_n = \frac{3(n-2)(n-3)(n-4)(n-5)}{4\cdot 4\cdot 6\cdot 6} b_n + \frac{2n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} a_n$$

$$e_n = -\frac{4(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} b_n$$

$$-\frac{3n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} a_n$$

etc.

Using Greek letters to distinguish terms, $\nabla^2\phi = 0$ is satisfied if:

$$\beta_n = -\frac{n(n-1)}{2\cdot 2} \alpha_n$$

$$\gamma_n = \frac{n(n-1)(n-2)(n-3)}{2\cdot 2\cdot 4\cdot 4} \alpha_n$$

$$\delta_n = \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} \alpha_n$$

etc.

The shear stress τ_{rz} at any point can therefore be represented as:

$$\tau_{rz} = [(-4 + 2\nu)\{(n-2)(n-3)b_n\} + (-1+\nu)\{\frac{n(n-1)(n-2)(n-3)}{2}\}a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2}\alpha_n]z^{n-4}r + [(3-\nu)\{\frac{(n-2)(n-3)(n-4)(n-5)}{4}\}b_n$$

$$+ (2-\nu)\{\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4}\}a_n$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4}\alpha_n]z^{n-6}r^3$$

$$+ [(-4 + \nu)\ \{\frac{(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{4\cdot 4\cdot 6}\}b_n$$

$$+ (-3 + \nu)\ \{\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{2\cdot 2\cdot 4\cdot 4\cdot 6}\}a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{2\cdot 2\cdot 4\cdot 4\cdot 6}\alpha_n]z^{n-8}r^5$$

etc.

The axial stress σ_z can also be represented at any point as:

$$\sigma_z = [(2-\nu)2\cdot 2(n-2)b_n + (1-\nu)n(n-1)(n-2)a_n$$

$$- n(n-1)(n-2)\alpha_n]z^{n-3}$$

$$+ [(-3+\nu)(n-2)(n-3)(n-4)b_n + (-2+\nu)\frac{n(n-1(n-2)(n-3)(n-4)}{2\cdot 2}a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2}\alpha_n]z^{n-5}r^2$$

$$+ [(4-\nu) \frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{4\cdot 4} b_n$$

$$+ (3-\nu) \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2\cdot 2\cdot 4\cdot 4} a_n$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2\cdot 2\cdot 4\cdot 4} \alpha_n] z^{n-7} r^4$$

etc.

The displacements u and w may also be represented at any point as:

$$u = \frac{1+\nu}{E} [\{-2(n-2)b_n + \frac{n(n-1)(n-2)}{2} \alpha_n\} z^{n-3} r$$

$$+\{\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2\cdot 4} a_n + \frac{2(n-2)(n-3)(n-4)}{4} b_n$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2\cdot 4} \alpha_n\} z^{n-5} r^3$$

$$+ \{- \frac{2n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2\cdot 2\cdot 4\cdot 4\cdot 6} a_n$$

$$- \frac{3(n-2)(n-3)(n-4)(n-5)(n-6)}{4\cdot 4\cdot 6} b_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2\cdot 2\cdot 4\cdot 4\cdot 6} \alpha_n \} z^{n-7} r^5] \qquad (6)$$

etc.

and

$$w = \frac{1+\nu}{E}[\{(1-\nu)n(n-1)a_n+(2-\nu)2\cdot 2b_n-n(n-1)\alpha_n\}z^{n-2}$$

$$+\{(-2+2\nu)\ \frac{n(n-1)(n-2)(n-3)}{2\cdot 2}\ a_n$$

$$+(-3+2\nu)(n-2)(n-3)b_n+\frac{n(n-1)(n-2)(n-3)}{2\cdot 2}\ \alpha_n\}z^{n-4}r^2$$

$$+\{(3-2\nu)\ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4\cdot 4}\ a_n$$

$$+(4-2\nu)\ \frac{(n-2)(n-3)(n-4)(n-5)}{4.4}\ b_n$$

$$-\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4\cdot 4}\ \alpha_n\}z^{n-6}r^4 \qquad (7)$$

etc.

These equations may therefore be utilized to form a series of polynomial solutions to equations (1) and (2). The ability of this form to produce a solution to the desired equilibrium and compatibility equations as well as the boundary conditions may therefore be easily investigated for any desired number of terms.

Considering a sum of general terms the shear stress at $z = \pm\frac{H}{2}$ becomes:

$$\tau_{rz} = \sum_n [(-4+2\nu)(n-2)(n-3)b_n + (-1+\nu)\frac{n(n-1)(n-2)}{2}\ a_n$$

$$+\frac{n(n-1)(n-2)(n-3)}{2}\ \alpha_n]\ (\pm\frac{H}{2})^{n-4}r$$

$$+ [(3-\nu)\ \frac{(n-2)(n-3)(n-4)(n-5)}{4}\ b_n$$

$$+ (2-\nu)\ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4}\ a_n$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 2\cdot 4}\ \alpha_n]\ (\pm\frac{H}{2})^{n-6} r^3$$

$$+ [(-4+\nu)\ \frac{(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{4\cdot 4\cdot 6}\ b_n$$

$$+ (-3+\nu)\ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{2\cdot 2\cdot 4\cdot 4\cdot 6}\ a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{2\cdot 2\cdot 4\cdot 4\cdot 6}\ \alpha_n]\ (\pm\frac{H}{2})^{n-8} r^5$$

etc.

In order to satisfy the first boundary condition, $\tau_{rz} = 0$ and therefore each order of r must be independent and must equal zero. It can also be seen that even values of n do not change sign for $z = \pm\frac{H}{2}$ but odd values of n do change sign. As a result, for each order of r the even and odd terms must be independent and must each equal zero. Expressing each sum in terms of its lowest order member we find:

For r

$$a_4 = \Sigma_{n=4,6,8\ldots} \frac{2(-4+2\nu)(n-2)(n-3)}{(1-\nu)\ 4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-4} b_n$$

$$+\Sigma_{n=4,6,8\ldots} \frac{n(n-1)(n-2)(n-3)}{(1-\nu)\ 4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-4} \alpha_n \tag{8}$$

$$+\Sigma_{n=6,8,10\ldots} \frac{-n(n-1)(n-2)(n-3)}{4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-6} a_n$$

$$a_5 = \Sigma_{n=5,7,9\ldots} \frac{2(-4+2\nu)(n-2)(n-3)}{(1-\nu)\ 5\cdot4\cdot3\cdot2} \left(\frac{H}{2}\right)^{n-5} b_n$$

$$+\Sigma_{n=5,7,9\ldots} \frac{n(n-1)(n-2)(n-3)}{(1-\nu)\ 5\cdot4\cdot3\cdot2} \left(\frac{H}{2}\right)^{n-5} \alpha_n \tag{9}$$

$$+\Sigma_{n=7,9,11\ldots} \frac{-n(n-1)(n-2)(n-3)}{5\cdot4\cdot3\cdot2} \left(\frac{H}{2}\right)^{n-7} a_n$$

For r^3

$$a_6 = \Sigma_{n=6,8,10\ldots} \frac{-2\cdot2(3-\nu)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 6\cdot5\cdot4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-6} b_n$$

$$+\Sigma_{n=6,8,10\ldots} \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 6\cdot5\cdot4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-6} \alpha_n \tag{10}$$

$$+\Sigma_{n=8,10,12\ldots} \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)}{6\cdot5\cdot4\cdot3\cdot2\cdot1} \left(\frac{H}{2}\right)^{n-8} a_n$$

$$a_7 = \Sigma_{n=7,9,11\ldots} \frac{-2\cdot 2(3-\nu)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-7} b_n$$

$$+\Sigma_{n=7,9,11\ldots} \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-7} \alpha_n \quad (11)$$

$$+\Sigma_{n=9,11,13\ldots} \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-9} a_n$$

For r^5

$$a_8 = \Sigma_{n=8,10,12\ldots} \frac{2\cdot 2(-4+\nu)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{(3-\nu)\ 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \left(\frac{H}{2}\right)^{n-8} b_n$$

$$+\Sigma_{n=8,10,12\ldots} \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{(3-\nu)\ 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \left(\frac{H}{2}\right)^{n-8} \alpha_n \quad (12)$$

$$+\Sigma_{n=10,12,14\ldots} \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \left(\frac{H}{2}\right)^{n-10} a_n$$

$$a_9 = \Sigma_{n=9,11,13\ldots} \frac{2\cdot 2(-4+\nu)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{(3-\nu)\ 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-9} b_n$$

$$+\Sigma_{n=9,11,13\ldots} \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{(3-\nu)\ 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-9} \alpha_n \quad (13)$$

$$+\Sigma_{n=11,13,15\ldots} \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \left(\frac{H}{2}\right)^{n-11} a_n$$

etc.

When higher order equivalents of the a terms are substituted into the lower order equivalents we find that each a_n can be expressed as a sum of equal and higher order terms of b and α. In the event that an a is zero, the same order b can be expressed as a sum of an equal order term of α and higher order terms of b and α. Similarly, if an a and b were both zero, the same order term of α can be expressed as a sum of higher order terms of b and α.

Disregarding the flat plate solution and attempting a general solution, the boundary conditions for σ_z become:

$$\sigma_z = 0 \text{ at } z = \frac{H}{2}$$

$$\sigma_z = -P \text{ at } z = -\frac{H}{2}$$

Consequently, the sums of σ_z terms containing r^2, r^4 etc. must all equal zero at $z = \pm\frac{H}{2}$ whereas the sums of terms which are functions only of z must be non-zero. The odd order terms do not change sign for $z = \frac{H}{2}$ or $z = -\frac{H}{2}$, however, the even order terms do. The odd and even sums containing r^2, r^4 etc. therefore each equal zero at $z = \pm\frac{H}{2}$. The sum of odd terms which is only a function of z must therefore be equal to $-\frac{P}{2}$ while the sum of even terms must be equal to $\frac{P}{2}$ at $z = \frac{H}{2}$ and $-\frac{P}{2}$ at $z = -\frac{H}{2}$. This results in the following equations:

$$\sum_{n=3,5,7} [(2-\nu)2\cdot 2(n-2)b_n + (1-\nu)n(n-1)(n-2)a_n$$

$$- n(n-1)(n-2)\ \alpha_n] \cdot (\tfrac{H}{2})^{n-3} = -\frac{P}{2}$$

$$\sum_{n=4,6,8..} [(2-\nu)2\cdot 2(n-2)b_n + (1-\nu)n(n-1)(n-2)a_n$$

$$- n(n-1)(n-2)\ \alpha_n]\ (\tfrac{H}{2})^{n-3} = \frac{P}{2}$$

For r^2

$$\sum_{n=5,7,9..} [(-3+\nu)(n-2)(n-3)(n-4)b_n + (-2+\nu)\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2}a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2}\ \alpha_n]\ (\tfrac{H}{2})^{n-5} = 0$$

$$\sum_{n=6,8,10...} [(-3+\nu)(n-2)(n-3)(n-4)b_n + (-2+\nu)\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2}a_n$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot 2}\ \alpha_n]\ (\tfrac{H}{2})^{n-5} = 0$$

etc.

If the values of "a" satisfying the shear stress boundary condition are substituted we find for a_3:

$$a_3 = \frac{-1}{2(1-\nu)\ 3\cdot2\cdot1} P + \frac{(-2+\nu)2}{(1-\nu)3} b_3 + \frac{1}{1-\nu} \alpha_3$$

$$+ \frac{2\cdot3\ [-2(-2+\nu)]}{(1-\nu)\ 3\cdot2\cdot1} \left(\frac{H}{2}\right)^2 b_5$$

$$+ \sum_{n=7,9,11..} \left\{\frac{2(n-2)[-2(n-4)(-2+\nu)]}{(1-\nu)\ 3\cdot2\cdot1} \left(\frac{H}{2}\right)^2\right.$$

$$\left. + \frac{-2(3-\nu)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 3\cdot2\cdot1\cdot2\cdot3}\right\} \left(\frac{H}{2}\right)^{n-5} b_n$$

$$+ \sum_{n=7,9,11..} \left\{\frac{n(n-1)(n-2)[2-(n-3)]}{(1-\nu)\ 3\cdot2\cdot1\cdot2} \left(\frac{H}{2}\right)^2\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(2-\nu)\ 3\cdot2\cdot1\cdot2\cdot3\cdot2}\right\} \left(\frac{H}{2}\right)^{n-5} \alpha_n$$

$$+ \sum_{n=9,11,13..} \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)}{3\cdot2\cdot1\cdot2\cdot3\cdot2} \left(\frac{H}{2}\right)^{n-7} a_n \qquad (14)$$

etc.

By substituting the appropriate orders of a, a_3 may be expressed in terms of P, b_3, α_3, b_5, and higher odd orders of b and α. The α_5 term does not appear directly in this expression although all orders of α do appear.

Similarly, we find an expression for fourth and higher orders of the r-independent sum:

$$(1-\nu)\ 6\cdot5\cdot4\ [(\tfrac{H}{2})^2 -1]\ (\tfrac{H}{2})\ a_6$$

$$+ \sum_{n=6,8,10...} 2(n-2)[2(2-\nu) + (-4+2\nu).(n-3)]\ (\tfrac{H}{2})^{n-3}\ b_n$$

$$+\sum_{n=6,8,10...} (-n)(n-1)(n-2)[(n-3)-1]\ (\tfrac{H}{2})^{n-3}\ \alpha_n$$

$$+ \sum_{n=8,10,12...} (1-\nu)n(n-1)(n-2)[(\tfrac{H}{2})^2 - 1]\ (\tfrac{H}{2})^{n-5}\ a_n = \frac{P}{2} \qquad (15)$$

In this boundary condition the shear stress boundary condition causes all fourth order terms to cancel resulting in an expression for sixth or higher order terms.

The odd sum for terms which are functions of r^2 becomes:

$$b_5 = \sum_{n=5,7,9...} \frac{-n(n-1)(n-2)(n-3)}{2\cdot2\cdot3}\left[\frac{(1-\nu)(n-4)+(-2+\nu)}{(-3+\nu)2(1-\nu)+(-2+\nu)(-4+2\nu)}\right](\tfrac{H}{2})^{n-5}\ \alpha_n$$

$$+\sum_{n=7,9,11...} \frac{-(n-2)(n-3)}{3\cdot2}\left[\frac{(-3+\nu)2(1-\nu)(n-4)+(-2+\nu)(-4+2\nu)}{(-3+\nu)2(1-\nu)+(-2+\nu)(-4+2\nu)}\right](\tfrac{H}{2})^{n-5}\ b_n$$

$$+\sum_{n=7,9,11} \frac{(2-\nu)n(n-1)(n-2)(n-3)(1-\nu)}{2\cdot2\cdot3[(-3+\nu)2(1-\nu)+(-2+\nu)(-4+2\nu)]}[(n-4)(\tfrac{H}{2})^2 -1]\ (\tfrac{H}{2})^{n-7}\ a_n$$

(16)

It is apparent by considering the higher order terms that none of these terms become zero. The even sum for terms which are functions of r^2 becomes:

$$b_8 = \sum_{n=8,10,12...} \frac{n(n-1)(n-2)(n-3)(n-4)}{6\cdot5\cdot4\cdot3\cdot2\cdot2}$$

$$\left[\frac{3(3-\nu)(n-6)+4(-2+\nu)(n-5)(n-6)(n-7)}{(-2+\nu)(-2)(-4+\nu)+2(3-\nu)(3-\nu)}\right]\left(\frac{H}{2}\right)^{n-8} \alpha_n$$

$$+\sum_{n=10,12,14...} \frac{-(n-2)(n-3)(n-4)(n-6)}{6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$\left[\frac{(3-\nu)^2 3\cdot2\cdot1+(-2+\nu)(-2)(-4+\nu)(n-5)(n-7)}{(-2+\nu)(-2)(-4+\nu)+2(3-\nu)(3-\nu)}\right]\left(\frac{H}{2}\right)^{n-8} b_n$$

$$+\sum_{n=10,12,14...} \frac{-(3-\nu)(-2+\nu)n(n-1)(n-2)(n-3)(n-4)}{6\cdot5\cdot4\cdot2\cdot2[(-2+\nu)(-2)(-4+\nu)+2(3-\nu)^2]}[1-(n-5)]\left(\frac{H}{2}\right)^{n-8} a_n$$

(17)

Once again the lowest order terms cancel. The same type behavior is found for terms which are functions of r^4, r^6, r^8, etc.

Several important characteristics appear in these equations. The odd order terms of fifth order or higher result in an expression of b in terms of the same and higher orders of α only. Since the third order terms did

not appear in the shear stress boundary condition, a_3 is expressed in terms of P, b_3, and α terms of third order and higher.

The even terms behave similarly to each other for each order of $\bar{z}$. In each case, the lowest order terms cancel each other and the next higher order of b can then be expressed in terms of the same and higher orders of α.

These derived expressions for the b terms can be substituted into the expressions from the first boundary condition. The final result for third order will be an expression for a_3 in terms of b_3, P, and third and higher orders of α. For fifth order and higher, the result will be expressions for a_n and b_n in terms of α of order n and higher. The final result for fourth order terms is an expression for a_4 in terms of b and α of fourth order and higher since the fourth order terms cancelled out of the final boundary condition. For sixth order terms, the final result is expressions for a_6 and b_6 in terms of P and α of order six and higher. For eighth and higher order even terms, the final result is expressions for a_n and b_n in terms of α of order n and higher. Consequently, the unevaluated constants available to satisfy the final boundary condition are:

$$b_3,\ \alpha_3,\ b_4,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8\ \cdots$$

The above-mentioned behavior of the equations satisfying the first boundary conditions combined with the previous attempted solution using third, fourth, and sixth order terms allows determination of the crucial factors which will or will not permit satisfaction of the final boundary condition. In the previous attempted solution equivalents of b_3, α_3, b_4, and α_4 appeared in the z^2, z, and constant terms of u and w. However, each term contained the same relationship between b_3 and α_3 and between b_4 and α_4. As a result, only 2 of these constants are useful in satisfying the final boundary condition. The cancellation of the sixth order terms in u and w combines with this to produce a total of five equations requiring satisfaction but only two constants with which to satisfy them. As a result, three additional constants are needed in order to satisfy the final boundary condition.

From equations (6), (7), (9), and (16) it can be seen that the addition of fifth order terms will introduce another constant α_5 to assist in satisfying the final boundary condition. In order to satisfy the final boundary condition, therefore, two higher order constants are needed.

Since the sixth order terms appear in equations (6) and (7) as α_6 and P and since the α_6 coefficients cancel, higher order terms are needed to cancel the P

multiples of z^3 and z^4. The next available terms are seventh and eighth order but equations (6) and (7) show that their introduction will cause u and w to have z^5 and z^6 components. The z^6 term will contain only α_8 and the z^5 term will contain only α_7 and α_8.

If α_7 and α_8 are utilized to eliminate the P multiples of z^3 and z^4, the z^5 and z^6 terms will be unsatisfied. The introduction of higher order terms will not solve this problem since equations (8) through (17) show that all higher order terms behave as the seventh and eighth order terms and the result would just be the raising of the order of the unsatisfied terms in the final boundary condition. The only possible way to completely satisfy the final boundary condition would require that the coefficients of two of the higher order terms cancel as the coefficients of α_6 cancelled.

Investigating this possibility for α_8 we find for the z^6 term in u:

$$u = \frac{1+\nu}{E} \left[- \frac{8 \cdot 7 \cdot 6}{2} 3(2-\nu) \alpha_8\right] z^6 \tan\theta$$

By examination of equations (8) through (17) it can be seen that none of the coefficients of α_n will cancel in u or w for $n \geq 7$. As a result it is impossible to satisfy all the boundary conditions for this problem with even an

infinite series of polynomials.

Since the possible values of z are all fairly small compared to the radius, the components of u and w at the boundary from the two unsatisfied highest orders should approach zero as the order increases. Consequently it would seem that by taking a sufficiently high order equation and satisfying the boundary conditions for all but a negligible value of u and w, an accurate approximation can be obtained.

This approach produces a sufficiently small error in the final boundary condition but when the resulting constants are used to calculate the deflection of the center of the viewport, it does not compare well to the experimental results obtained by Stachiw. This occurs because although only the high order boundary condition is unsatisfied, the high order term affects the evaluation of all lower order constants. This is apparent from equation (14). Consequently, this method cannot even provide a reasonable approximation.

DISCUSSION OF RESULTS

A solution form for a viewport in a rigid mount is proposed which appears to have an ample number of terms to be able to satisfy all boundary conditions. The compatibility equations are such that the satisfaction of the first two boundary conditions couples the available terms so that the final boundary condition cannot be satisfied. Although for high orders the error is small, the terms are coupled in such a way that all terms are affected.

The inability of this approach to provide satisfactory results is due to the compatibility conditions and by the assumption of plane stress. The nature of the equations with plane stress is such that it cannot accommodate a slanting surface for the final boundary condition.

Consequently, even for thin viewports plane stress cannot be assumed in forming an analytic solution. The general three-dimensional compatibility equations must be utilized and the simplification of stress functions cannot be used.

The equilibrium and compatibility equations can best be solved by expressing them in terms of u and w. The

equilibrium equations become:

$$\left(\frac{2}{1-2\nu} G-\lambda\right) \frac{\delta}{\delta r} \left(\frac{\delta u}{\delta r} + \frac{u}{r}\right) + G \frac{\delta^2 u}{\delta z^2} + (G+\lambda) \frac{\delta^2 w}{\delta r \delta z} = 0$$

and

$$(G+\lambda) \frac{\delta}{\delta z} \left(\frac{\delta u}{\delta r} + \frac{u}{r}\right) + G\left(\frac{\delta^2 w}{\delta r^2} + \frac{1}{r} \frac{\delta w}{\delta r}\right) + \left(\frac{2}{1-2\nu} G - \lambda\right) \frac{\delta^2 w}{\delta z^2} = 0$$

The compatability equations become:

$$\lambda \frac{(1-\nu)}{\nu} \left(\frac{\delta^3 u}{\delta r^3} + \frac{1}{r} \frac{\delta^2 u}{\delta r^2} + \frac{\delta^3 u}{\delta r \delta z^2}\right.$$

$$+ \lambda \left(\frac{1}{r} \frac{\delta^2 u}{\delta r^2} - \frac{1}{r^2} \frac{\delta u}{\delta r} + \frac{u}{r^3} + \frac{\delta^3 w}{\delta r^2 \delta z} + \frac{1}{r} \frac{\delta^2 w}{\delta r \delta z} + \frac{1}{r} \frac{\delta^2 u}{\delta z^2} + \frac{\delta^3 w}{\delta z^3}\right)$$

$$- \frac{4G}{r^2} \left(\frac{\delta u}{\delta r} - \frac{u}{r}\right) + \frac{\lambda}{\nu} \left(\frac{1}{r} \frac{\delta^2 u}{\delta r^2} - \frac{2}{r^3} \frac{\delta u}{\delta r} + \frac{2u}{r^3} + \frac{\delta^3 u}{\delta r^3} + \frac{\delta^3 w}{\delta r^2 \delta z}\right. = 0$$

$$\lambda \frac{1-\nu}{\nu} \left(\frac{1}{r} \frac{\delta^2 u}{\delta r^2} - \frac{1}{r^2} \frac{\delta u}{\delta r} + \frac{u}{r^3} + \frac{1}{r} \frac{\delta^2 u}{\delta z^2}\right)$$

$$+ \lambda \left(\frac{\delta^3 u}{\delta r^3} + \frac{1}{r} \frac{\delta^2 u}{\delta r^2} + \frac{\delta^2 u}{\delta r \delta z} + \frac{\delta^3 w}{\delta r^2 \delta z} + \frac{1}{r} \frac{\delta^2 w}{\delta r \delta z} + \frac{\delta^3 w}{\delta z^3}\right)$$

$$+ \frac{4G}{r^2} \left(\frac{\delta u}{\delta r} - \frac{u}{r}\right) + \frac{\lambda}{\nu} \left(\frac{1}{r} \frac{\delta^2 u}{\delta r^2} + \frac{1}{r^2} \frac{\delta u}{\delta r} - \frac{u}{r^3} + \frac{1}{r} \frac{\delta^2 w}{\delta r \delta z}\right) = 0$$

$$\lambda \frac{1-\nu}{\nu} \left(\frac{\delta^3 w}{\delta r^2 \delta z} + \frac{1}{r} \frac{\delta^2 w}{\delta r \delta z} + \frac{\delta^3 w}{\delta z^3}\right)$$

$$+\lambda \left(\frac{2}{r} \frac{\delta^2 u}{\delta r^2} - \frac{1}{r^2} \frac{\delta u}{\delta r} + \frac{u}{r^3} + \frac{\delta^3 u}{\delta r^3} + \frac{1}{r} \frac{\delta^2 u}{\delta r \delta z} + \frac{\delta^3 u}{\delta r \delta z^2}\right)$$

$$+ \frac{\lambda}{\nu} \left(\frac{1}{r} \frac{\delta^2 u}{\delta z^2} + \frac{\delta^3 u}{\delta r \delta z^2} + \frac{\delta^3 w}{\delta z^3}\right) = 0$$

$$G \left[2 \frac{\delta^3 u}{\delta_r{}^2 \delta_z} + \frac{\delta^3 u}{\delta z^3} + \frac{2}{r} \frac{\delta^2 u}{\delta r \delta z} - \frac{2}{r^2} \frac{\delta u}{\delta z}\right.$$

$$\left. + \frac{\delta^3 w}{\delta r^3} + 2 \frac{\delta^3 w}{\delta r \delta z^2} - \frac{1}{r^2} \frac{\delta w}{\delta r}\right] = 0$$

Once a solution form is found which satisfies these equations, the boundary conditions can be utilized to evaluate the constants in the equation.

RECOMMENDATIONS

The solution of the aforementioned equilibrium and compatibility equations will not be simple. A preliminary investigation reveals that no simple assumptions provide solutions. If no analytic solution is obtained, designs can still be safely made using numerical techniques but an analytic solution would prove much more useful to others. An analytic solution would also provide a great deal of insight into the stresses in the reinforced area of the hull which serves as a mounting. For these reasons, further work should be done to try to achieve a solution to this problem.

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